Exercise 37

Find the limit or show that it does not exist.

$$\lim_{x\to\infty} \frac{1-e^x}{1+2e^x}$$

Solution

Make the substitution, $u = e^x$, so that as $x \to \infty$, $u \to \infty$. Then multiply the numerator and denominator by the reciprocal of the highest power of u in the denominator.

$$\begin{aligned}
\lim_{x\to\infty} \frac{1-e^x}{1+2e^x} &= \lim_{u\to\infty} \frac{1-u}{1+2u} \\
&= \lim_{u\to\infty} \frac{1-u}{1+2u} \cdot \frac{\frac{1}{u}}{\frac{1}{u}} \\
&= \lim_{u\to\infty} \frac{(1-u)\frac{1}{u}}{(1+2u)\frac{1}{u}} \\
&= \lim_{u\to\infty} \frac{\frac{1}{u}-1}{\frac{1}{u}+2} \\
&= \frac{\lim\limits_{u\to\infty}\left(\frac{1}{u}-1\right)}{\lim\limits_{u\to\infty}\left(\frac{1}{u}+2\right)} \\
&= \frac{\lim\limits_{u\to\infty}\frac{1}{u} - \lim\limits_{u\to\infty} 1}{\lim\limits_{u\to\infty}\frac{1}{u} + \lim\limits_{u\to\infty} 2} \\
&= \frac{0-1}{0+2} \\
&= -\frac{1}{2}
\end{aligned}$$

Alternatively, without a substitution,

$$\lim_{x\to\infty} \frac{1-e^x}{1+2e^x} = \lim_{x\to\infty} \frac{1-e^x}{1+2e^x} \cdot \frac{e^{-x}}{e^{-x}} = \lim_{x\to\infty} \frac{e^{-x}-1}{e^{-x}+2} = \frac{0-1}{0+2} = -\frac{1}{2}.$$